Rotating Sources and the Kerr Geometry

Consider starting w/ a round (spherically symmetric) object and then spinning it (giving it nonzero angular momentum J about an axis through its center.

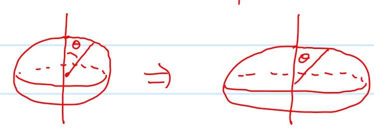
GR ~ 1915 → 1 year later (1916)

The original object would induce a Schwarzschild geometry $ds^2 = -(1-\frac{2GM}{r})dt^2 + (1-\frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2$

The spinning object will induce a different geometry first described by Kerr.

Two differences to anticipate:

- Spinning (at a constant rate) will still be time-independent (static), but not $t \to -t$ symmetric (stationary) so we should expect $dt\,dx^i$ cross-terms.
- If the spin axis is aligned w/ the poles, then we should expect θ-dependence from the "squashing" of the sphere, i.e.

  Note: we still have ϕ-independence

  Note: this is not cylindrical (no translation invariance along a "z"-axis)

After some soul searching (1963):

$$ds^2_{Kerr} = -\left(1-\frac{2GMr}{\rho^2}\right)dt^2 - \frac{2GMar\sin^2\theta}{\rho^2}(2\,d\phi\,dt) + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]d\phi^2$$

where:

$\{t, r, \theta, \phi\}$ Boyer-Lindquist coordinates

$$a = \frac{L_\phi}{M}$$

$$\Delta(r) = r^2 - 2GMr + a^2$$

$$\rho^2(r,\theta) = r^2 + a^2\cos^2\theta$$

To note:

- For $a \to 0$ $ds^2 \to ds^2_{Schwarzschild}$
- For $r \to \infty$ (w/ M, a fixed) $\overbrace{ds^2 \to -dt^2 + dr^2 + r^2 d\Omega^2}^{\mathbb{M}^4}$ (Asymptotic flatness)
- For $M \to 0$ (w/ a fixed) $ds^2 = -dt^2 + \frac{(r^2+a^2\cos^2\theta)}{(r^2+a^2)}dr^2 + (r^2+a^2\cos^2\theta)d\theta^2 + (r^2+a^2)\sin^2\theta\, d\phi^2$

  which is really just $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

  w/ $x = \sqrt{r^2+a^2}\,\sin\theta\cos\phi$

  $y = \sqrt{r^2+a^2}\,\sin\theta\sin\phi$

  $z = r\cos\theta$

  } Oblate spheroidal coordinates

- $d\phi\,dt$ cross-term might have been expected since it is rotating in ϕ

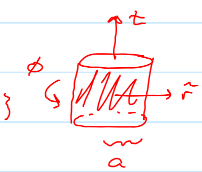
We notice that the metric is badly behaved when $\rho = 0$ and when $\Delta = 0$. Let's investigate these in turn.

$\rho = 0$ is a true curvature singularity (a curvature scalar blows up), but it is an interesting one.

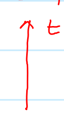
$\rho^2 = r^2 + a^2\cos^2\theta = 0$ requires both $r = 0$ and $\theta = \frac{\pi}{2}$. Contrast w/ Schwarzschild where $r = 0$ blows up for any θ or ϕ.

Before we look at $\rho = 0$, let's actually look at $r = 0$ (which turns out to be nontrivial)

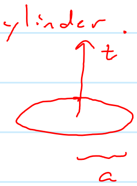
$$ds^2\Big|_{r=0} = -dt^2 + \underbrace{(a\cos\theta\, d\theta)^2 + (a\sin\theta)^2 d\phi^2}_{d\tilde{r}^2 + \tilde{r}^2 d\phi^2} \quad \text{w/ } \tilde{r} = a\sin\theta \quad \theta \in [0, \pi] \Rightarrow \tilde{r} \in [0, a]$$



So $r = 0$ is actually a 2+1 D volume which is cylindrical in $\{t, \tilde{r}, \phi\}$
Compare to Schwarzschild where $ds^2\big|_{r=0} = -dt^2$, i.e. a line along t



Okay, that's weird, but getting back to $\rho = 0$, we need $\theta = \frac{\pi}{2}$ which is @ $\tilde{r} = a$, i.e. the outer edge of the cylinder. So our singularity is spatially a "ring" which is then extended in time.



Note: The region inside of the ring is nonsingular. You could pass through it w/out dying!

Should this be expected? Well yes, sort of. We know that our source had 2 parameters, M and J_ϕ. If the singularity was pointlike, we could encode M, but what about J_ϕ? Now we see that J_ϕ is encoded in the deformation of the point to a ring, analogous to the deformation of the sphere to the spinning oblate spheroid. In fact the ring is extended around ϕ.

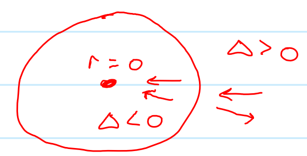
You may have guessed that J_ϕ could be encoded in the "intrinsic spin" of the singularity, but GR is classical and doesn't know about that nonsense!

Okay, what about $\Delta = 0$? First of all it is not a curvature singularity. In fact, much like w/ Schwarzschild, it indicates the presence of the horizon.

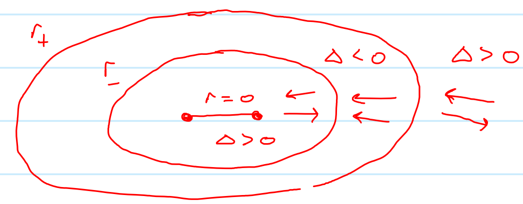
But: $\Delta(r) = r^2 - 2GMr + a^2 = 0 \Rightarrow r_\pm = GM \pm \sqrt{G^2M^2 - a^2}$ (2 horizons)

Let's vary a and see what happens. But first, note that the sign of Δ determines the sign in front of dr^2. For $\Delta > 0$, r is spatial and can increase or decrease, but if $\Delta < 0$, r is timelike so only moves in one direction (compare to Schwarzschild case).

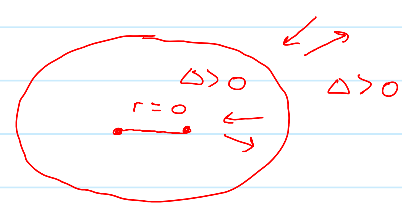
- $a = 0$ $\quad r_- = 0$ (the Schwarzschild singularity)
  $\quad r_+ = 2GM$ (the Schwarzschild horizon)



- $a^2 < G^2M^2$ "sub-extremal" $\quad r_+ > r_- > 0$



- $a^2 = G^2M^2$ "extremal" $\quad r_+ = r_- = GM$



- $a^2 > G^2M^2$ "over-extreme" w/ no horizon $\Rightarrow$ a naked singularity

Wednesday, April 19, 2017 7:06 PM

To maximally extend the Kerr geometry, we could adopt Kruskal-type coordinates, but let's go ahead and set up a more powerful tool.

Conformal (or Penrose) diagrams:

$\mathbb{M}^{4}$ $ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$ $\quad t \in (-\infty, \infty),\ r \in [0, \infty)$

Let's grab infinity and snuggle up to it w/

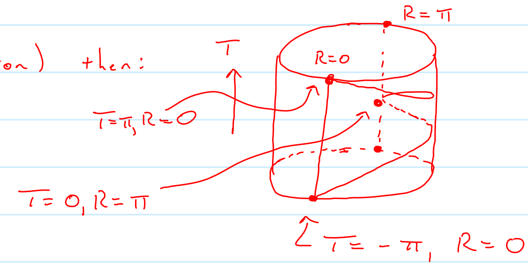
$$\left.\begin{aligned} T &= \tan^{-1}(t+r) + \tan^{-1}(t-r) \\ R &= \tan^{-1}(t+r) - \tan^{-1}(t-r) \end{aligned}\right. \Rightarrow \left.\begin{aligned} 0 \le R < \pi \\ |T| < \pi - R \end{aligned}\right\} \text{both have finite ranges}$$

Then: $ds^2 = \dfrac{1}{(\cos T + \cos R)^2}\left(-dT^2 + \underbrace{dR^2 + \sin^2 R\, d\Omega^2}_{\text{Note } R \text{ is acting like an angle}}\right)$

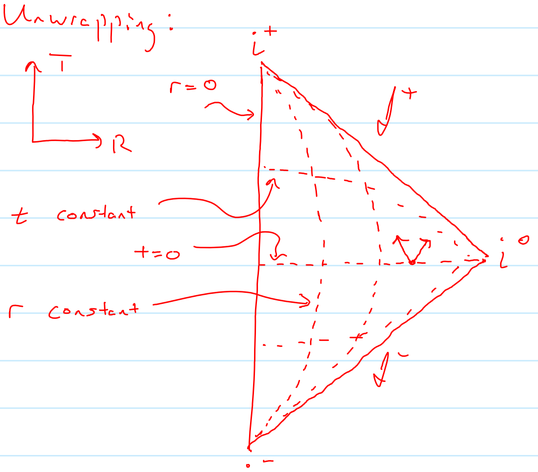
or

$ds^2 = \dfrac{1}{\omega^2(T,R)}\, \underbrace{d\tilde{s}^2}_{\text{conformally related geometry}}$ (preserves angles, in particular light-cones!!)

To visualize it, suppress θ + ϕ (purely radial motion) then:



Unwrapping:



i^+ = timelike future ∞ } All timelike geodesics ($m>0$) begin and end here
i^- = timelike past ∞

i^0 = spacelike ∞ All spacelike geodesics begin and end here

$\mathcal{J}^+$ = future null ∞ } All lightlike geodesics ($m=0$) begin and end here
$\mathcal{J}^-$ = past null ∞